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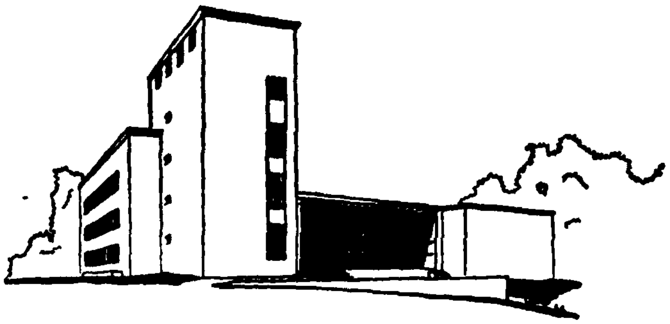

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A PROCEDURE FOR GENERATING THE CONVEX HULL OF A 0-1 PROGRAMMING POLYTOPE WITH POSITIVE COEFFICIENTS.

by

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Abstract

We present a procedure which generates all the facets of a 0-1 programming polytope P with positive coefficients in a finite number of steps. The procedure is based upon the relationship between facets of P and facets of the knapsack polytopes corresponding to certain nonnegative combinations of inequalities implied by P. Finiteness of the procedure is proven by examining the relationship of the valid inequalities generated during each step of the procedure in connection with a result due to Chvátal [7]. In addition to exploring the properties of inequalities generated by the procedure, several properties of the classes of valid inequalities for the knapsack polytope defined in Balas [1] and Balas and Jeroslow [3] are presented. In particular, the set of canonical inequalities [3] is shown to belong to Chvátal's elementary closure.

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1. Introduction

Consider the 0-1 programming problem

$$\max \sum_{j\in N} c_j x_j$$

(MKP)

$$\sum_{j\in N} a^i_j x_j \leq a^i_0 \text{ , } i \in M = \{1,2,\ldots,m\} \quad (1)$$

$$x_j = 0 \text{ or } 1, \quad j \in N, \quad (2)$$

where $a^i_j \geq 0$, $j \in N \cup \{0\}$, $i \in M$. This problem is often referred to as the multidimensional knapsack problem or as a monotone 0-1 program. Such problems arise in many useful mathematical programming models such as the discrete capital budgeting problem.

We shall find it useful to also represent (MKP) in the matrix form $\max \ cx$

$$Ax \leq A_0$$

$$x_j = 0 \text{ or } 1, \ j \in N.$$

Also denote the i^{th} inequality of (1) as $a^i x \leq a^i_0$.

The set

$$P = P(A,A_0) = \text{conv}\{x \in R^n | Ax \leq A_0, \ x_j = 0 \text{ or } 1, \ j \in N\}$$

will be called the <u>multidimensional knapsack polytope</u>

corresponding to the system $Ax \leq A_0$. The inequality

$$\sum_{j \in N} a_j x_j \leq a_0, \tag{3}$$

where $a_j \geq 0$, $j \in N$, is referred to as a knapsack inequality and is sometimes denoted as $ax \leq a_0$ or simply (a,a_0). The set $P(a,a_0)$ is called its corresponding <u>knapsack</u> <u>polytope</u>.

An inequality $\alpha x \leq \alpha_0$ is called a facet of P if i) $\alpha x \leq \alpha_0$ is a valid inequality for P, i.e. $\alpha x \leq \alpha_0$ is satisfied for all $x \in P$, and ii) there exist d affinely independent points of P which satisfy $\alpha x = \alpha_0$, where d is the dimension of P. It is well known that $d = |N|$, i.e. P is full-dimensional, if and only if $a_j^i \leq a_0^i$, for all $j \in N$, $i \in M$. Unless otherwise stated, we shall assume P to be full-dimensional.

Clearly the advantage of explicitly knowing the set $\mathcal{F}(P)$ of all facets of P is the fact that (MKP) can be solved as a linear program. Since the number of such facets is typically at least exponential, one perhaps ought not aspire to the goal of obtaining them all in a practical setting. However, a finite procedure for generating these facets or perhaps strong valid inequalities for P could be useful in the sense that the constraint set $Ax \leq A_0$ could be amended to include some of these valid inequalities in the hope of closing the gap between the optimal value of (MKP) and the value of the linear programming relaxation.

It is natural to seek the relationship between the facets of the multidimensional knapsack polytope P and the facets of the individual knapsack polytopes $P(a^i,a_0^i)$ corresponding to each constraint in (1). In general, P has facets which are not facets of any $P(a^i, a_0^i)$, $i \in M$, and many facets of the $P(a^i, a_0^i)$ are not facets of P. Knapsack inequalities and facets of their corresponding polytopes have received considerable attention in the literature [1],[2],[3],[4],[5],[9],[12],[13], and [14]. This paper deals with the task of generating $\mathcal{F}(P)$ by exploring this relationship.

Balas and Zemel [5] have shown that any facet of P can be obtained by a sequence of operations which involves the complementing of certain variables, the application of a generalized lifting procedure, and a subsequent recomplementing of the relevant variables. (See also Wolsey [13].) This process can be viewed as a finite procedure for generating all the facets of P; however, the generalized lifting procedure involves the solution of many smaller multidimensional knapsack problems. When specialized to a 0-1 knapsack problem, the systematic application of this sequence of operations yields a finite procedure for generating all the facets of the knapsack polytope, and it then only requires the solution of many 0-1 knapsack problems.

As previously suggested, we are exploring the relationship between the facets of the multidimensional knapsack polytope P and those of certain knapsack polytopes which are derived from the constraints of (1). We shall therefore assume that given any knapsack inequality, we can generate the set of all facets of its corresponding knapsack polytope in a finite number of steps by applying a procedure such as the one noted above.

An iterative procedure for generating the facets of P is presented in the next section. After showing that each iteration (step) of the procedure requires a finite number of operations, we show that the procedure will indeed generate all the facets of P in a finite number of steps. This is accomplished by examining the inequalities generated during each iteration in relation to a result due to Chvátal [7]. In the course of exploring this relationship, some properties concerning the family of valid inequalities for the knapsack polytope defined in [1] are stated and proved. In particular, the set of canonical inequalities [3] are shown to belong to Chvátal's elementary closure.

2. A Finite Procedure for Generating $\mathcal{F}(P)$

In this section we describe a procedure for generating the set $\mathcal{F}(P)$ of all facets of P. The procedure is shown to be complete in the sense that it generates all facets of P, and it is also shown to be finite. We first establish some necessary definitions.

Given the set of inequalities (1), a subset $K \subseteq N$ is said to be a cover for (1) if

$$\sum_{j\in K} a_j^i > a_0^i \tag{4}$$

for some $i \in M$.

The cover K is said to be minimal if

$$\sum_{j\in T} a_j^i \leq a_0^i, \qquad i \in M \tag{5}$$

for all proper subsets T of K.

Obviously, if K is a minimal cover for (1), then it is a minimal cover for $a^i x \leq a_0^i$ for at least one $i \in M$.

For any point $x \in \{0,1\}^n$, where $n = |N|$, define supp(x) $= \{j \in N | x_j = 1\}$. As previously mentioned, an inequality $ax \leq a_0$ is said to be valid for the polytope P if $ax \leq a_0$ is satisfied for all $x \in P$.

An inequality $ax \leq a_0$ is said to dominate the inequality $bx \leq b_0$, where a_0 and b_0 are not both 0, if there exists a real number $\lambda > 0$ such that

$$\lambda a_0 \leq b_0$$

$$\lambda a_j \geq b_j, \text{ for all } j \in N.$$

If in addition $\lambda a_k > \lambda b_k$ for some $k \in N$, then $ax \leq a_0$ is said to strictly dominate $bx \leq b_0$.

We now introduce a second notion of dominance. An inequality A is said to c-dominate an inequality B if every 0-1 point satisfying A also satisfies B. Additionally, if there exists a 0-1 point x satisfying B but not A, then A strictly c-dominates B.

Clearly, c-dominance is a weaker property than dominance. We now present an example which shows that it is strictly weaker.

Example 1. Consider the inequalities

$$2x_1 + x_2 + x_3 + x_4 \leq 2 \tag{6}$$

and

$$x_1 + x_2 \leq 1. \tag{7}$$

Inequality (6) does not dominate inequality (7); yet, it c-dominates it.

This notion of c-dominance will play an important role in defining the procedure to follow. We now state some results which should provide some insight into why such a notion is desirable.

Proposition 1. The inequality

$$\sum_{j\in N} a_j x_j \le a_0 \qquad (3)$$

c-dominates

$$\sum_{j\in N} b_j x_j \le b_0 \qquad (8)$$

if and only if every cover M for (8) is also a cover for (3).

Proof. Suppose there exists a cover C for (8) which is not a cover for (3), and let $\tilde{x} \in \{0,1\}^n$ be defined by $\mathrm{supp}(\tilde{x}) = C$. Since C is a cover for (8) but not for (3), we have that $\tilde{x}$ satisfies (3) but not (8), and hence (3) does not c-dominate (8).

Conversely, suppose that (3) does not c-dominate (8). Then there exists a 0-1 point $\tilde{x}$ such that $a\tilde{x} \le a_0$ but $b\tilde{x} > b_0$. Let $C = \mathrm{supp}(\tilde{x})$. Obviously, C is then a cover for (8) but not for (3). ||

We therefore see why the term c-dominate was chosen. The following corollary follows directly from the definition of c-dominance and that of the knapsack polytope $P(a,a_0)$.

Corollary 1.1. The inequality $ax \le a_0$ c-dominates $bx \le b_0$ if and only if $P(a,a_0) \subseteq P(b,b_0)$.

Corollary 1.2. If the inequality $ax \le a_0$ c-dominates $bx \le b_0$ and if $\beta x \le \beta_0$ is a valid inequality for $P(b,b_0)$, then it is a valid inequality for $P(a,a_0)$.

The proof follows immediately from the fact that $P(a,a_0) \subseteq P(b,b_0)$ if $ax \le a_0$ c-dominates $bx \le b_0$.

The next result concerns a relationship between some of the facets of two different knapsack polytopes corresponding to valid inequalities for P.

<u>Proposition 2.</u> Let $ax \leq a_0$ and $bx \leq b_0$ both be valid inequalities for the multidimensional knapsack polytope P. If $ax \leq a_0$ c-dominates $bx \leq b_0$, and if some facet $\beta x \leq \beta_0$ of $P(b,b_0)$ is also a facet of P, then $\beta x \leq \beta_0$ is a facet of $P(a,a_0)$.

<u>Proof.</u> Since $ax \leq a_0$ and $bx \leq b_0$ are both valid inequalities for P, and since $ax \leq a_0$ c-dominates $bx \leq b_0$, we have $P \subseteq P(a,a_0) \subseteq P(b,b_0)$. From the preceding corollary it follows that $\beta x \leq \beta_0$ is a valid inequality for $P(a,a_0)$. Since it is a facet of P, there exist n affinely independent points of P, and hence of $P(a,a_0)$ which satisfy $\beta x = \beta_0$. Therefore $\beta x \leq \beta_0$ is a facet of $P(a,a_0)$. ||

Let T be an arbitrary, finite set of knapsack inequalities given by

$$\sum_{j \in N} b_j^i x_j \leq b_0^i, \qquad i = 1,\ldots,t, \qquad (9)_T$$

and let

$$F(T) = \bigcup_{(b^i,b_0^i) \in T} \mathcal{F}(P(b^i,b_0^i)),$$

where for any polytope Q, $\mathcal{F}(Q)$ is the set of all its facets.

Also, let

$$C(T) = \{bx \leq b_0 \mid (b,b_0) = \sum_{i=1}^{t} \lambda_i (b^i,b_0^i),\ \lambda_i \geq 0, (b^i,b_0^i) \in T\},$$

i.e. C(T) is the set of all nonnegative combinations of inequalities in T.

A set $R \subseteq C(T)$ is said to <u>c-represent</u> C(T) if given any inequality $(b,b_0) \in C(T)$, there exists an inequality $(a,a_0) \in R$ such that (a,a_0) c-dominates (b,b_0). Equivalently, R is also said to be a <u>c-representation</u> of C(T). Notice that C(T) is a c-representation of itself.

By definition, the set $C(T)$ is infinite. However, every inequality $(b,b_0) \in C(T)$ corresponds to a unique set of covers, i.e. subsets of N. Since the number of such distinct sets of subsets is finite, we have just proven the following proposition.

<u>Proposition 3.</u> There exist c-representations of $C(T)$ which are finite in cardinality.

Appropriately, such sets shall be called <u>finite c-representations</u> of $C(T)$. We next state a result which will allow us to actually construct a finite c-representation of $C(T)$.

<u>Proposition 4.</u> Let $K = \{C_1, C_2, \ldots, C_p\}$ be a set of covers for $(9)_T$. There exists $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_t)$ with $\lambda_i \geq 0$, $i = 1,\ldots,t$, such that each set C_ℓ, $\ell = 1,\ldots,p$ is a cover for the inequality

$$\sum_{j\in N} \left(\sum_{i=1}^{t} \lambda_i b_j^i \right) x_j \leq \sum_{i=1}^{t} \lambda_i b_0^i \tag{10$_\lambda$}$$

if and only if there exists a solution $y_i, i = 1,\ldots,t$ to the system

$$\sum_{i=1}^{t} \gamma_i^\ell \, y_i > 0, \qquad \ell = 1,\ldots,p, \tag{11$_K$}$$

where for each $\ell = 1,\ldots,p$ and $i = 1,\ldots,t$

$$\gamma_i^\ell = \sum_{j\in C_\ell} b_j^i - b_0^i. \tag{12}$$

In fact, any solution y_i, $i = 1,\ldots,t$ will yield such a λ by setting $\lambda_i = y_i$, $i = 1,\ldots,t$.

<u>Proof.</u> Suppose there exists a nonnegative vector λ such that each set C_ℓ, $\ell = 1,\ldots,p$, covers $(10)_\lambda$. We then have

$$\sum_{j\in C_\ell} \left(\sum_{i=1}^{t} \lambda_i b_j^i \right) > \sum_{i=1}^{t} \lambda_i b_0^i, \qquad \ell = 1,\ldots,p \tag{13}$$

or equivalently

$$\sum_{i=1}^{t} \left(\sum_{j \in C_\ell} b_j^i - b_0^i \right) \lambda_i > 0, \qquad \ell = 1,\dots,p. \tag{14}$$

Therefore, in light of (12) we see that $(11)_K$ is satisfied by $y_i = \lambda_i$, $i = 1,\dots,t$.

Conversely, if $(11)_K$ is satisfied for some y_i, $i = 1,\dots,t$, then setting $\lambda_i = y_i$, $i = 1,\dots,t$, we observe that the λ_i, $i = 1,\dots,t$ satisfy (14) and hence they also satisfy (13). Consequently, each set C_ℓ, $\ell = 1,\dots,p$ will be a cover for $(10)_\lambda$. ||

We can thus construct a finite c-representation $\hat{G}$ of C(T) as follows:

(α) Enumerate the set $\mathcal{C}$ of all covers for $(9)_T$, and then generate $\mathcal{K} = \mathcal{P}(\mathcal{C}) \setminus \{\emptyset\}$, where $\mathcal{P}(\mathcal{C})$ is the power set (set of all subsets) of $\mathcal{C}$.

(β) Starting with $\hat{G} = \emptyset$, for each $K \in \mathcal{K}$ decide whether the system $(11)_K$ is consistent. If $(11)_K$ is inconsistent, go on to the next set $K \in \mathcal{K}$. Otherwise, choose any <u>one</u> solution λ to $(11)_K$, add the inequality $(10)_\lambda$ to $\hat{G}$, and continue on to the next K.

This algorithm will clearly generate a finite c-representation $\hat{G}$ of C(T) in a finite number of steps. Although any finite c-representation G would suffice within the framework of the procedure for generating $\mathcal{F}(P)$ which is soon to be defined, it turns out that we can further restrict our attention to certain proper subsets of G.

For any finite c-representation G of C(T), any subset M(T) of G satisfying both

i) $M(T) = \{(a,a_0) \in G \mid$ there does not exist $(b,b_0) \in G$ such that (b,b_0) strictly c-dominates $(a,a_0)\}$, and

ii) if (a^λ,a_0^λ), (a^θ,a_0^θ) both belong to $M(T)$ and (a^λ,a_0^λ) c-dominates (a^θ,a_0^θ), then $\lambda = \theta$,

is said to be a _minimal complete set of nonnegative combinations of inequalities in T_.

We immediately remark that $M(S)$ is not uniquely defined since it depends upon both the particular finite c-representation G from which it is extracted and its arbitrary selection from among all those subsets of G satisfying properties i) and ii) of the definition. Therefore, when we say $M(T)$, we mean some arbitrary but fixed $M(T)$.

It should also be observed that if we use the previously defined algorithm for generating a finite c-representation $\hat{G}$ of $C(T)$, then since property ii) is satisfied by all inequalities in $\hat{G}$ (by construction), $\hat{G}$ will give rise to a unique set $M(T)$. This $M(T)$ can be identified in a finite number of steps by a process which uses a binary representation of $\mathcal{P}(\mathcal{C})$ (i.e. for each set $K \in \mathcal{P}(\mathcal{C})$, the corresponding binary vector will have a 1 as the j^{th} component if and only if the j^{th} cover of $\mathcal{C}$ belongs to K) and then selects those inequalities whose corresponding set of covers are maximal with respect to the lexicographic ordering of the binary vectors.

We are now ready to state the procedure for generating all the facets of P.

A Finite Procedure for Generating $\mathcal{F}(P)$

Step 0. Let S_0 denote the set (1) of original knapsack inequalities, and let $F^0(S_0) = S_0$.

Step 1. a) Generate $M(S_0)$.

b) Generate $F^1(S_0) = F(M(S_0))$.

Step k. a) Generate $M(S_{k-1})$, where $S_{k-1} = F^{k-1}(S_0)$.

b) Generate $F^k(S_0) = F(M(S_{k-1}))$.

It should be clear from the previous discussion that each step of the procedure requires a finite number of operations. By utilizing a result due to Chvátal in [7], we show that there exists a nonnegative integer p such that $\mathcal{F}(P) \subseteq F^p(S_0)$. The smallest such integer p shall be called the f-rank of S_0. We first state a definition and a theorem from [7] for reference. Let U be the general set of linear inequalities

$$Cx \leq C_0.$$

Definition. (Chvátal [7]) An inequality $ax \leq a_0$ belongs to the elementary closure of U, denoted $e^1(U)$, if there exists $\lambda \geq 0$ such that

$$\lambda C = a = \text{integer}$$
$$[\lambda C_0] \leq a_0,$$

where $[r]$ is the greatest integer less than or equal to r. Further, let $e^k(U) = e^1(e^{k-1}(U))$.

Theorem 1. (Chvátal [7]) There exists an integer q such that

$$\mathcal{J}(P(C,C_0)) \subseteq e^q(U).$$

The smallest such integer q is called the rank of U in [7], however, we shall refer to it as the k-rank of U.

Letting S be the set of knapsack inequalities (1) and S* be S together with the constraints $0 \leq x_j \leq 1$, $j \in N$, we are now ready to state and prove the following result.

Theorem 2. For the set S of knapsack inequalities, if for some integer $h > 0$, the inequality $bx \leq b_0$ belongs to $e^h(S^*)$, then there exists an inequality $ax \leq a_0$ in $M(F^{h-1}(S))$ such that $ax \leq a_0$ c-dominates $bx \leq b_0$. In particular, if $bx \leq b_0$ is a facet of P, then $bx \leq b_0$ belongs to $F^h(S)$.

Proof. We prove the theorem by induction on h. If $bx \leq b_0$ belongs to $e^1(S^*)$, then there exist nonnegative multipliers λ_i, i = 1, 2, ..., m = |M|, δ_j and γ_j, j = 1, 2, ..., n = |N|, such that

$$\sum_{i=1}^{m} \lambda_i a_j^i + \delta_j - \gamma_j = b_j, \quad \text{for all } j \in N$$

and

$$[b_0^*] \leq b_0,$$

where

$$b_0^* = \sum_{i=1}^{m} \lambda_i a_0^i + \sum_{j=1}^{n} \delta_j.$$

Consider the inequality $\sum_{j \in N} \hat{b}_j x_j \leq \hat{b}_0$ given by

$$\hat{b}_j = \sum_{i=1}^{m} \lambda_i a_j^i, \quad \text{for all } j \in N \cup \{0\}.$$

Clearly $\hat{b}x \leq \hat{b}_0$ belongs to C(S), and also $\hat{b}x \leq \hat{b}_0$ c-dominates

$bx \leq b_0^*$, which in turn c-dominates $bx \leq b_0$. Since M(S) is a (minimal) complete set of positive combinations of inequalities in S, there exists an inequality $ax \leq a_0$ belonging to $M(S) = M(F^0(S))$ which c-dominates $\hat{b}x \leq \hat{b}_0$ and thus $bx \leq b_0$ also.

Now, inductively assume that the result holds for $e^{h-1}(S^*)$, where h is some positive integer. If $bx \leq b_0$ belongs to $e^h(S^*)$, then there exists a set of t inequalities in $e^{h-1}(S^*)$ given by

$$c^k x \leq c_0^k, \quad k = 1, 2, \ldots, t,$$

and nonnegative multipliers θ_k, $k = 1, 2, \ldots, t$ such that

$$\sum_{k=1}^{t} \theta_k c_j^k = b_j, \quad \text{for all } j \in N,$$

and

$$\left[\sum_{k=1}^{t} \theta_k c_0^k\right] \leq b_0.$$

From the induction hypothesis, for each $k = 1, 2, \ldots, t$, there exists an inequality

$$d^k x \leq d_0^k$$

belonging to $M(F^{h-2}(S))$ such that $d^k x \leq d_0^k$ c-dominates $c^k x \leq c_0^k$. Let $\mathcal{F}_k = \mathcal{F}(P(d^k, d_0^k)) = \{\sum_{j \in N} f_{ij}^k x_j \leq f_{i0}^k \mid i = 1, 2, \ldots, \ell(k)\}$ be the set of facets defining the knapsack polytopes

$$P(d^k, d_0^k) = \text{conv}\{x \mid d^k x \leq d_0^k,\ x_j = 0 \text{ or } 1,\ j \in N\}$$

for each $k = 1, 2, \ldots, t$.

Since $P(d^k, d_0^k) \subseteq P(c^k, c_0^k)$ for each $k = 1, 2, \ldots, t$, there exist multipliers σ_i^k, $i = 1, 2, \ldots, \ell(k) = |\mathcal{F}_k|$ satisfying

$$\sum_{i=1}^{\ell(k)} \sigma_i^k f_{ij}^k = c_j^k, \quad \text{for all } j \in N,$$

and

$$\sum_{i=1}^{\ell(k)} \sigma_i^k f_{i0}^k \leq c_0^k .$$

As before, $bx \leq b_0$ is c-dominated by

$$\sum_{j\in N} \left(\sum_{k=0}^{t} \theta_k c_j^k \right) x_j \leq \sum_{k=0}^{t} \theta_k c_0^k. \tag{15}$$

By substitution, (15) is dominated by

$$\sum_{j\in N} \left(\sum_{k=0}^{t} \sum_{i=1}^{\ell(k)} \theta_k \sigma_i^k f_{ij}^k \right) x_j \leq \sum_{k=0}^{t} \sum_{i=1}^{\ell(k)} \theta_k \sigma_i^k f_{i0}^k, \tag{16}$$

which is a nonnegative combination of inequalities in $F^{h-1}(S)$. By definition, there exists an inequality $ax \leq a_0$ belonging to $M(F^{h-1}(S))$ such that $ax \leq a_0$ c-dominates (16) and hence $bx \leq b_0$.

The last assertion of the theorem follows directly from Proposition 3, and the fact that if $bx \leq b_0$ is a facet of P, then it is also a facet of $P(b,b_0)$. ||

It immediately follows from this theorem and Theorem 1 that the previously stated procedure will indeed generate all the facets of P in a finite number of steps. We also see that the f-rank of S is less than or equal to the k-rank of S*.

Obviously this procedure requires a large number of operations to generate $\mathcal{F}(P)$. It is therefore natural to investigate the nature of those inequalities obtained after one pass of the procedure. In particular, we shall compare such inequalities to those belonging to the elementary closure of S*. In order to do this, we call $F^1(S)$ the <u>elementary f-closure</u> of S and henceforth refer to $e^1(S^*)$ as the <u>elementary k-closure</u> of S*. These two elementary closures are the next topic of discussion.

3. Properties of the Elementary Closures.

Recall that we have already established that the f-rank of S is less than or equal to the k-rank of S*, where S is the set of knapsack inequalities (1) and S^* is S together with the constraints $0 \leq x_j \leq 1$, $j \in N$. In the same manner, it follows from Theorem 2 that any facet contained in the elementary k-closure of S^* will also be obtained in the elementary f-closure of S. We shall characterize those inequalities and in particular, those facets of P which are obtainable in the elementary k-closure of S^*. We then apply this characterization to an example showing that certain facets of P belonging to the elementary f-closure of S can not be obtained in the elementary k-closure of S^*. We conclude from this that the elementary f-closure of S generally contains more facets of the multidimensional knapsack polytope P than does the elementary k-closure of S^*.

Once again let U be the general set of linear inequalities

$$Cx \leq C_0.$$

We now characterize those inequalities which belong to $e^1(U)$ and note that the sufficiency of condition (17) is given in [7].

<u>Proposition 5</u>. The inequality $\alpha x \leq \alpha_0$ belongs to $e^1(U)$ if and only if

$$\max \{\alpha x \mid Cx \leq C_0\} < \alpha_0 + 1. \tag{17}$$

<u>Proof</u>. Condition (17) holds if and only if the system of linear inequalities

$$Cx \leq C_0$$
$$\alpha x \geq \alpha_0 + 1$$

is inconsistent. This system is inconsistent if and only if the following system is consistent (see, for example, [11])

$$\lambda C = \alpha$$
$$\lambda C_0 < \alpha_0 + 1$$
$$\lambda \geq 0,$$

which is true if and only if $\alpha x \leq \alpha_0$ belongs to $e^1(U)$. ||

Observe that the testing of condition (17) merely involves the solution of a linear program.

This proposition is now used in the following example which shows that a facet of the multidimensional knapsack polytope P can belong to the elementary f-closure of S without belonging to the elementary k-closure of S*.

Example 2. Consider the set T consisting of the two knapsack inequalities

$$7x_1 + 6x_2 + 4x_3 + 3x_4 + 2x_5 + 2x_6 + x_7 + x_8 \leq 7 \quad (18)$$

$$2x_1 + x_2 + 2x_3 + x_4 + x_5 + x_6 + x_7 + x_8 \leq 3. \quad (19)$$

Let $\hat{P}$ be the multidimensional knapsack polytope corresponding to the system defined by T. Adding these two inequalities together, we obtain

$$9x_1 + 7x_2 + 6x_3 + 4x_4 + 3x_5 + 3x_6 + 2x_7 + 2x_8 \leq 10 \quad (20)$$

which belongs to C(T). Now, the inequality

$$2x_1 + x_2 + x_3 + x_5 + x_7 \leq 2 \quad (21)$$

is easily shown to be both a facet of the knapsack polytope corresponding to (20) and also of $\hat{P}$. Therefore, (21) belongs to the elementary f-closure of T. However, it can not belong to the elementary k-closure of T*(i.e. T together with $0 \leq x_j \leq 1$, $j = 1, 2, \ldots, 8$) since

$$\max\{2x_1 + x_2 + x_3 + x_5 + x_7 | x \text{ satisfies both (18) and (19)}, \; 0 \leq x_j \leq 1, \; j = 1,2,\ldots,8\} = 3.$$

We thus see that the elementary f-closure generally contains more facets of P than does the elementary k-closure. The next example shows that the elementary f-closure unfortunately does not contain all the facets of P.

Example 3. Consider the set W of inequalities

$$6x_1 + 3x_2 + x_3 + 3x_4 + x_5 \leq 6 \tag{22}$$

$$x_1 + 3x_2 + 3x_3 + x_4 + 3x_5 \leq 8, \tag{23}$$

and let $\tilde{P}$ denote the corresponding multidimensional knapsack polytope. The inequality

$$3x_1 + 2x_2 + x_3 + x_4 + x_5 \leq 3 \tag{24}$$

is a facet of $\tilde{P}$. However, inequality (24) is not valid for either of the knapsack polytopes P_1, P_2 corresponding to (22) and (23), respectively. Nor is it a valid inequality for any knapsack polytope arising from a nonnegative combination of (22) and (23). This is easily seen if one considers that there is no nonnegative combination of (22) and (23) for which the sets $M_1 = \{2, 3, 4\}$ and $M_2 = \{2, 3, 5\}$ are both covers; whereas, M_1 and M_2 both cover (24). Therefore, inequality (24) does not belong to $F^1(W)$.

We conclude our discussion by presenting some properties of the classes of valid inequalities derived from minimal covers for the knapsack polytope $P(a, a_0)$ which were defined in [1] and [3]. Recall that for any minimal cover M of the knapsack inequality (3), the set $E(M) = M \cup M'$, where

$$M' = \{j \in N \setminus M \mid a_j \geq a_{j_1}\}$$

and

$$a_{j_1} = \max_{j \in M} \{a_j\},$$

is called the extension of M to N. Letting $\mathcal{M}$ be the set of all minimal covers for (3), Balas and Jeroslow have shown in [3] that inequality (3) is equivalent to the set of canonical inequalities

$$\sum_{j \in E(M)} x_j \leq |M| - 1, \text{ for all } M \in \mathcal{M}. \qquad (25)_M$$

In [1], Balas defines a family of strong valid inequalities for $P(a,a_0)$, many of which are facets, based on the following result. Assume $P(a,a_0)$ is full-dimensional.

Theorem 3. (Balas [1]) The inequality

$$\sum_{j \in N} \beta_j x_j \leq \beta_0 \qquad (26)$$

is a valid inequality for $P(a,a_0)$ if $\beta_0 = |M|-1$ for some minimal cover M for (3), $\beta_j = 0$ for all $j \in N \setminus E(M)$, and for $j \in E(M)$, $\beta_j = h$, where h is the uniquely defined integer satisfying

$$\sum_{i \in M_h} a_i \leq a_j < \sum_{i \in M_{h+1}} a_i, \qquad (27)$$

where M_h is the set of the first h elements of M, $h = 1, \ldots, |M|$.

If in addition one has

$$\sum_{i \in M \setminus M_{h+1}} a_i + a_j \leq a_0, \text{ for all } j \in N_h, \quad h = 0, 1, \ldots, |M|,$$

where

$$N_h = \{j \in N | \beta_j = h\}, \ h = 0, 1, \ldots, |M|,$$

then (26) is a facet of $P(a,a_0)$.

Observe that inequality (21) of Example 2 belongs to this class of valid inequalities for the knapsack polytope corresponding to inequality (20).

Letting R* be the set of linear inequalities

$$ax \leq a_0$$

$$0 \leq x_j \leq 1, \quad j \in N,$$

the following result provides a necessary and sufficient condition for the inequality $\beta x \leq \beta_0$ derived from the application of the preceding theorem to some minimal cover for (3) not to belong to the elementary k-closure of R*.

<u>Proposition 6</u>. The inequality $\beta x \leq \beta_0$ belonging to the class of valid inequalities for $P(a, a_0)$ defined in Theorem 3 does not belong to $e^1(R^*)$ if and only if there exists an index $h \in \{j \in N | \beta_j \geq 2\}$ with

$$\frac{\beta_h}{a_h} (a_0 - \sum_{j \in W^{\beta_0}} a_j) \geq 1, \tag{28}$$

where $W = \{j \in N | \beta_j = 1\} = \{j_1, j_2, \ldots, j_{|W|}\}$ is written such that $a_{j_1} \geq a_{j_2} \geq \ldots \geq a_{j_{|W|}}$, and W^k is the last k elements in W.

<u>Proof</u>. The inequality $\beta x \leq \beta_0$ can be obtained by applying Theorem 3 to the knapsack inequality $ax \leq a_0$ with the minimal cover $M = W^{\beta_0+1}$. Suppose that such an index h exists. Consider the fractional point $\tilde{x}$ defined by

$$\tilde{x}_j = \begin{cases} 1 & \text{if } j \in W^{\beta_0}, \\ \dfrac{a_0 - \sum_{j \in W^{\beta_0}} a_j}{a_h} & \text{if } j = h, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly $a\tilde{x} = a_0$ by construction. Also, since

$$a_h + \sum_{j \in W^{\beta_0}} a_j \geq \sum_{j \in M} a_j > a_0 ,$$

we have

$$\tilde{x}_h = \frac{a_0 - \sum_{j \in W^{\beta_0}} a_j}{a_h} < 1.$$

(Observe that this implies that condition (28) can hold only if $\beta_h > 1$.) Now,

$$\beta\tilde{x} = \sum_{j \in W^{\beta_0}} \beta_j + \beta_h \tilde{x}_h \geq \beta_0 + 1,$$

since $\beta_j = 1$, for all $j \in W$ and condition (28) holds for h. By Proposition 5 we have $\beta x \leq \beta_0$ does not belong to $e^1(R^*)$.

Conversely, suppose that such an index $h \in \{j \in N | \beta_j \geq 2\}$ satisfying (28) does not exist. It is well known that an optimal solution x^0 to the continuous knapsack problem

$$\max \{z = \beta x | ax \leq a_0,\ 0 \leq x_j \leq 1,\ j \in N\} \tag{29}$$

can be obtained by reordering the indices in N as $\{i_1, i_2, \ldots, i_n\}$,

where $\dfrac{\beta_{i_1}}{a_{i_1}} \geq \dfrac{\beta_{i_2}}{a_{i_2}} \geq \cdots \geq \dfrac{\beta_{i_n}}{a_{i_n}}$,

and defining

$$x_j^0 = \begin{cases} 1 & \text{if } j \in \{i_1, i_2, \ldots, i_{k-1}\} \\ \dfrac{a_0 - \sum_{j=i_1}^{i_{k-1}} a_j}{a_{i_k}} & \text{if } j = i_k, \\ 0 & \text{otherwise,} \end{cases}$$

where $\sum_{t=1}^{k-1} a_{i_t} \leq a_0 < \sum_{t=1}^{k} a_{i_t}$.

Clearly, both $x_{i_k}^0 < 1$ and $\beta_{i_k} > 0$. Now, since $\beta x \leq \beta_0$ is a valid inequality for $P(a,a_0)$, it follows that $\sum_{t=1}^{k-1} \beta_{i_t} \leq \beta_0$. In fact $\sum_{t=1}^{k-1} \beta_{i_t} = \beta_0$, for suppose $\sum_{t=1}^{k-1} \beta_{i_t} \leq \beta_0 - 1$. Then $\beta x^0 < (\beta_0 - 1)+1 = \beta_0$, but consider the point $x^* \in \{0,1\}^n$ defined by $\mathrm{supp}(x^*) = W^{\beta_0}$. We then have that $\beta x^* = \beta_0 > \beta x^0$ which contradicts the optimality of x^0 for (29). We next consider two cases which are collectively exhaustive.

Case 1. If $\beta_{i_k} = 1$, then

$$z = \beta x^0 = \sum_{t=1}^{k-1} \beta_{i_t} + x^0_{i_k} < \beta_0 + 1$$

and $\beta x \leq \beta_0$ belongs to $e^1(R^*)$.

Case 2. If $\beta_{i_k} \geq 2$, then $\sum_{t=1}^{k-1} a_{i_t} \geq \sum_{j \in W^{\beta_0}} a_j$ since $\sum_{p=1}^{k-1} \beta_{i_p} = \beta_0$ and the β_{i_p}, $p = 1, \ldots, k-1$ are each defined by condition (27). Therefore,

$$z = \beta x^0 = \beta_0 + \beta_{i_k} \left(\frac{a_0 - \sum_{p=1}^{k-1} a_{i_k}}{a_{i_k}}\right)$$

$$\leq \beta_0 + \beta_{i_k} \left(\frac{a_0 - \sum_{j \in W^{\beta_0}} a_j}{a_{i_k}}\right)$$

$$< \beta_0 + 1$$

since condition (28) does not hold for any $j \in N$ such that $\beta_j \geq 2$. Thus, $\beta x \leq \beta_0$ belongs to $e^1(R^*)$. ||

The following consequence of Proposition 6 is immediate.

Corollary 6.1. All canonical inequalities [3] implied by $ax \leq a_0$ belong to the elementary closure of R^*.

We now illustrate Proposition 6 in the following example.

Example 4. Let P^* be the knapsack polytope corresponding to the inequality

$$9x_1 + 7x_2 + 6x_3 + 4x_4 + 3x_5 + 3x_6 + 2x_7 + 2x_8 + x_9 \leq 10 \quad (30)$$

taken from [1]. The inequality

$$2x_1 + x_2 + x_3 + x_5 + x_6 \leq 2 \quad (31)$$

can be derived by the application of Theorem 3 to the minimal cover {3, 5, 6} for (30), and it belongs to the elementary k-closure of

(30) together with the constraints $0 \leq x_j \leq 1$, $j = 1, 2, \ldots, 9$ since

$$\frac{2}{9}(10 - (3+3)) = \frac{8}{9} < 1.$$

However, the inequality

$$2x_1 + x_2 + x_3 \qquad + x_5 \qquad + x_7 \qquad\qquad \leq 2$$

derived from the minimal cover $\{3, 5, 7\}$ does not since

$$\frac{2}{9}(10 - (3+2)) = \frac{10}{9} > 1.$$

Although we have characterized those inequalities and hence, those facets of $P(a, a_0)$ belonging to the family of valid inequalities defined by Theorem 3 which belong to $e^1(R^*)$, it is not correct to assume that these are the only type of facets belonging to $e^1(R^*)$. For instance, consider the following.

Example 5. The inequality

$$2x_1 + 2x_2 + 2x_3 + 2x_4 + x_5 + x_6 + x_7 \leq 4 \tag{32}$$

is a facet of the knapsack polytope corresponding to

$$3x_1 + 3x_2 + 3x_3 + 2x_4 + 2x_5 + 2x_6 + 2x_7 \leq 6. \tag{33}$$

Using Proposition 5, we can easily determine that (32) belongs to the elementary k-closure of (33) together with the constraints $0 \leq x_j \leq 1$, $j = 1, 2, \ldots, 7$. However, the facet (32) does not belong to the class of valid inequalities for the knapsack polytope corresponding to inequality (33) defined by Theorem 3. In fact, as shown in [4], the facet (32) can not be sequentially lifted from any minimal cover of (33).

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We present a procedure which generates all the facets of a 0-1 programming polytope P with positive coefficients in a finite number of steps. The procedure is based upon the relationship between facets of P and facets of the knapsack polytopes corresponding to certain nonnegative combinations of inequalities implied by P. Finiteness of the procedure is proven by examining the relationship of the valid inequalities generated during each step of the procedure in connection with a result due to Chvátal [7]. In (cont'd)

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addition to exploring the properties of inequalities generated by the procedure, several properties of the classes of valid inequalities for the knapsack polytope defined in Balas [1] and Balas and Jeroslow [3] are presented. In particular, the set of canonical inequalities [3] is shown to belong to Chvátal's elementary closure.

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